Estimating Sensor Lifetime using an Event Based Control Strategy

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ABSTRACT

We study the problem of estimating the sensor lifetime following an event based control strategy. The sensor is used to observe a continuous process and the measurement produced by the sensor is sent to a base station only if the observed parameter deviates from the model by a given threshold. Using this event based control strategy, we are able to show that the expect lifetime of the sensor grows quadratically with the size of the monitoring region, and is inversely proportional to the square of the diffusion coefficients of the process and measurement noises. Simulations are provided to verify the theory developed.

General Terms

Event based control, Dynkin's formula, stopping times, sensor network

Keywords

Estimation, Event based control

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1. INTRODUCTION

Wireless sensor networks consist of large number of small devices, capable of some limited computation, communication and sensing, operating under energy constraints, see [9]. These networks are intended for a broad range of environmental sensing applications from weather data-collection to target tracking and habitat monitoring, see [2] [3].

Energy efficiency is one of the primary goals in these networks, because networks operate on small inexpensive batteries with limited lifetimes. It has been documented that sensors spend most of their energy during communication, and the amount of energy consumed for sensing and data processing is considered negligible, see [7]. This suggests power management strategies for sensor communication which depend on the problem at hand. For example in target tracking or intrusion detection where great accuracy is needed to localize and then intercept a target, then sensors such as radars are required to produce measurements on a regular time basis. Other systems instead are required to respond only to rare events. For example, diagnostic systems in modern buildings employ sensors and actuators to monitor air temperatures, room temperatures, air pressures, etc.. over time, and issue alerts when values exhibit a significant deviation from normal operating conditions. This requires the design of an architecture where sensors only transmit when a specific event occurs, thereby obtain an *event-based* control scheme, see [6]. The scheme is illustrated in Figure 1. The plot shows a model that predicts a parameter of interest, in this case the temperature of an environment, as a function of time. Associated with this model is a tolerance band: a binary signal is only sent by the monitoring sensor when the temperature crosses this tolerance band. The binary information is used to discriminate whether the upper or lower part of the band was crossed.

The main contribution of our paper is an analytical formula to characterize the average lifetime of a monitoring sensor on the basis of the system parameters. The formula can be used for energy efficient design and sensitivity analysis. The rest of the paper is organized as follows. Section 2

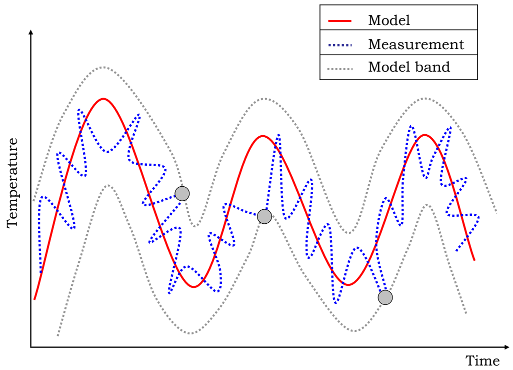

Figure 1: Event based control scheme using a single sensor monitoring the temperature of an environment.

presents the general setup, the problem of interests and introduces the mathematical framework for our problem. Section 3 contains the main result. Section 4 presents numerical simulations to further support the validity of the analytical findings. Section 5 concludes the paper. More proofs and results are provided in the appendices.

2. GENERAL SETUP

2.1 Model Definition

We begin by describing the model of process and measurement noise used in this paper. We consider two independent noise sequences:

- The *process noise* at time t is defined to be $f(t)-g(t)$, where $f(t)$ is the true value of a parameter at time t and $g(t)$ is the value predicted for that parameter by the model at time t.
- The *measurement noise* at time t is the difference $m(t)-f(t)$, where $m(t)$ is the observed value at time t for the parameter.

A signal is generated by the sensor when the measurement deviates from the current model by more than a given threshold γ, i.e. when

$$|m(t)-g(t)|>\gamma \tag{1}$$

Thus a signal is generated when the observed value (process noise plus measurement noise) exceeds the threshold. Since we assume that measurements are being taken continuously, we model both measurement noise and model noise as continuous time stochastic processes.

In habitat monitoring applications, measurements are produced by an instrument that consists of redundant digital transducers and Wiener (Brownian) noise is usually the dominant noise component, see [1]. Therefore, we use the Wiener process to model the measurement noise. Let us denote by $X_t^{(1)}$ the measurement noise at time t. Then its dynamics is described by the stochastic differential equation

$$dX_t^{(1)}=\alpha\cdot dW_t^{(1)} \tag{2}$$

with $W_t^{(1)}$ being a one-dimensional Wiener process, and α being the diffusion constant.

We choose to model the process noise using an Ornstein-Uhlenbeck (OU) process [4], a mean-reverting gaussian process which experiences a drift towards the initial value of magnitude proportional to its displacement. This is because if the used model is accurate, then we expect the process noise to always revert towards the reversion level zero. In mathematical terms, an OU process is the solution of the stochastic differential equation

$$dX_t^{(2)}=-\kappa X_t^{(2)}dt+\sigma dW_t^{(2)} \tag{3}$$

where $W_t^{(2)}$ is a one-dimensional Wiener process, and κ and σ are positive constants, representing respectively the mean reversion speed and the diffusion level.

In our study we assume that $W_t^{(1)}$ and $W_t^{(2)}$ are independent.

2.2 Problems of interest

We present a study case of a network consisting of two sensors; the first sensor also referred to as the *monitoring sensor* is employed to measure a parameter of interest in the outside environment (such as pressure or temperature) and the second sensor, also referred to as the *base station* is used to perform the estimation of the parameter. In order to minimize the amount of energy expenditure, we consider that the two sensors share a model and that a binary signal is only sent by the monitoring to the base station when the parameter of interest deviates from the model by more than a specified threshold. All this raises two interesting questions:

(1) *Performance characterization of sensor lifetime*: how do we characterize the average lifetime of the monitoring sensor, knowing that it will only communicate to the base station when a significant model deviation occur?

(2) *Constrained estimation*: How does the base station update the estimate of the parameter using the received measurements and the random times at which these are sent?

Due to space limitations, in this paper we only address the first question, and leave the second question open for future research.

2.3 Mathematical Formulation

Equation (1) can be reformulated as

$$|X_t^{(1)}+X_t^{(2)}|>\gamma \tag{4}$$

where $X_t^{(1)}$ and $X_t^{(2)}$ have been defined respectively in eq.(2) and eq.(3). The inequality (4) defines a region where the deviation of the measurement from the model is smaller than γ. We denote this area by Ω_0. The boundary $\partial\Omega_0$ consists of two parallel lines denoted by Γ_{-1} (left line) and Γ_1 (right line). When the measurement exceeds the model prediction by γ, then the monitoring sensor sends a binary signal to the base station. The new model used will be centered around either Γ_{-1} or Γ_1 depending on the hit boundary. The deviation from the model will be of the same size γ. We propose the following mathematical formulation to represent the set of model deviations used over time:

$$\Omega_k = \{(x_1, x_2) : |x_1 + x_2 - k\gamma| < \gamma\}, \qquad k \in \mathbb{Z} \quad (5)$$

Equation (5) splits the plane into overlapping regions. Similarly to the base case above, the boundary $\partial\Omega_k$ consists of two parallel lines that belong to Ω_{k-1} and Ω_{k+1} denoted by Γ_{k-1} and Γ_{k+1} respectively. An illustrative example including few regions is presented in Figure 2.

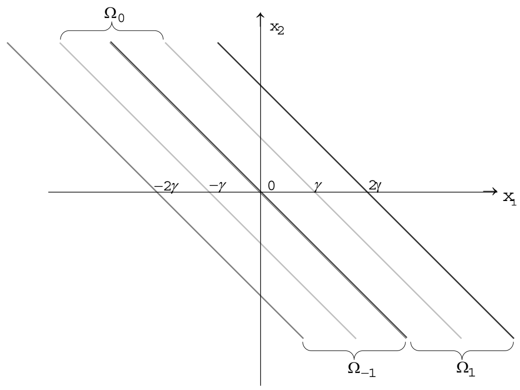

Figure 2: Regions Ω_{-1}, Ω_0 and Ω_1

3. MAIN RESULT

We provide an analytical characterization of the average lifetime of the sensor battery, given an initial energy level ε, and an energy expenditure e required for each communication. Let us denote by

$$M = \frac{\varepsilon}{e} \quad (6)$$

the number of communication periods before the sensor runs out of battery. Let τ_i denotes the i-th hitting time of the boundary of some region. The random variable L denoting the sensor lifetime is defined as

$$L = \sum_{i=1}^{M} \tau_i \quad (7)$$

The main result of the paper is stated in the following theorem

THEOREM 1. *The average lifetime of the sensor is given by:*

$$E[L] = M \frac{\gamma^2}{\alpha^2(1+\sigma^2)} \quad (8)$$

In order to prove the theorem, we first prove the lemma

LEMMA 1. *Assume $(X_0^{(1)}, X_0^{(2)}) = (x, -x)$. The expected waiting time before the first signal is sent is given by*

$$E[\tau_1] = \frac{\gamma^2}{\alpha^2(1+\sigma^2)} \quad (9)$$

PROOF. Using Dynkin's Lemma stated in Appendix B, we have that

$$E[f(X_{\tau_1}^{(1)}, X_{\tau_1}^{(2)})] = f(x, -x) + E\left[\int_0^{\tau_1} \widehat{\mathcal{L}} f(X_s^{(1)}, X_s^{(2)}) ds\right] \quad (10)$$

where $\widehat{\mathcal{L}}$ is the generator of the diffusion process associated to eq.(2) and eq.(3), and f is an arbitrary smooth function. The detailed development of the generator is given in Appendix A. Here, only the final expression is reported

$$\widehat{\mathcal{L}} := -\kappa x_2 \partial_{x_2} + \frac{1}{2}\left(\alpha^2 \partial^2_{x_1 x_1} + \sigma^2 \partial^2_{x_2 x_2}\right) \quad (11)$$

In order to use eq.(10) to compute the expected inter-arrival time, we choose a function f such that

(1) $f(X_{\tau_1}^{(1)}, X_{\tau_1}^{(2)})$ does not depend on τ_1, and it is easy to compute.

(2) $\hat{\mathcal{L}} f$ is easy to compute and the result is a simple function of τ_1, so that the integral in eq.(10) is easily to evaluate.

In order to satisfy both (1) and (2) we choose a function f which is identically zero on the boundary of the region and such that $\hat{\mathcal{L}} f = 1$, so that the integral in eq.(10) is just τ_1. Then, eq.(10) would reduce to

$$E[\tau_1] = -f(x, -x) \quad (12)$$

Therefore, the problem is reduced to finding the solution of the ordinary differential equation with Dirichlet zero-boundary conditions given by

$$\begin{aligned} \hat{\mathcal{L}} f &= 1 \\ f(x_1, x_2) &= 0, \qquad |x_1 + x_2| = \gamma \end{aligned} \quad (13)$$

Using regular perturbation theory, we are able to derive an approximate solution of eq.(13). The detailed development is presented in Appendix C. Here, we only state the final result given by

$$f(x_1, x_2) = \frac{(x_1 + x_2)^2}{\alpha^2(1+\sigma^2)} - \frac{\gamma^2}{\alpha^2(1+\sigma^2)} \quad (14)$$

The statement of the lemma follows immediately from eq.(14) since $f(x, -x) = -\frac{\gamma^2}{\alpha^2(1+\sigma^2)}$. □

We now proceed to prove the main theorem.

PROOF. Since our process is Markovian, the inter-arrival time between two consecutive hits is independent of the region. Therefore, the random variables $\tau_1, \tau_2, \ldots, \tau_k$ are statistically identical. The proof of the theorem 1 follows from the linearity of the expectation and the above observation as detailed next

$$\begin{aligned} E[L] &= E[\sum_{i=1}^{M} \tau_i] \\ &= \sum_{i=1}^{M} E[\tau_i] \\ &= M E[\tau_1] \\ &= M \frac{\gamma^2}{a^2(1+\sigma^2)} \end{aligned} \quad (15)$$

□

Equation (15) shows that the time needed before the sensor battery is exhausted grows quadratically with the size of the monitoring region γ and it is inversely proportional to the square of diffusion coefficients σ and α of the process and measurement noise. From a theoretical point of view, this makes intuitive sense because larger γ means larger monitoring regions and therefore longer time is needed to hit their boundaries. Moreover, large diffusion coefficients make the process visit a larger portion of the state space in a smaller amount of time, thus increasing the time needed to touch the boundary of the region. Formula (15) has also implications from a design point of view. Larger accuracy of measuring devices (small α) impacts favorably the sensor lifetime, namely increasing it by a quadratic factor. The diffusion constant σ of the model noise reflects the accuracy of the model used. If a predictive model that tracks measurements with great accuracy is used, this will result in the model noise process having a large coefficient for the mean reversion speed κ and a small diffusion coefficient σ. According to our formula (15) this will result in a quadratic gain in the average sensor lifetime.

The choice of the size γ of the monitoring region requires designers to make a tradeoff between estimation accuracy and energy consumption. Large values of γ increase the average sensor lifetime by a quadratic factor, but at the same time increase the estimation error of the true state due to the smaller number of generated signals. An exact characterization of the relation between the estimation error and the size γ of the region is left for future research.

4. NUMERICAL SIMULATIONS

The objective of this section is to test the estimation power of our formula in estimating the average time of interarrival between consecutive signals.

4.1 The simulation Method

We simulate the Brownian paths as the limit of a random walk, i.e. using the update equation

$$X^{(1)}(t+\Delta T) = X^{(1)}(t) + \alpha\sqrt{\Delta T}N(0,1) \qquad (16)$$

where $N(0,1)$ is a standard gaussian random variable. An exact analytical update formula for the Ornstein-Uhlenbeck process is known [5]. We simulate the sample path of the process $X^{(2)}(t)$ in eq.(3) using the update equation

$$X^{(2)}(t+\Delta t) = X^{(2)}(t)\mu + \sigma_X N(0,1) \qquad (17)$$

where

$$\mu = e^{-\kappa\Delta t}, \qquad \sigma_X^2 = \frac{\sigma^2}{2\kappa}(1-\mu^2) \qquad (18)$$

We set the time discretization ΔT to 0.01 sec.

4.2 Expected Sensor Lifetime: Numerical versus Estimated

We conduct a series of experiments to support the validity of the theoretical derivations given in Section 3. To make the simulations realistic, we consider Berkeley Mica motes which have been used for habitat monitoring applications, see [3]. Mica runs on a pair of AA batteries, which have been estimated to supply 2200 mAh at 3 volts. The cost of a packet transmission is 20.000nAh. Assuming that the energy is only dissipated through packet transmission, we obtain that a number of 110 transmissions can be executed before the mote runs out of battery. Following their study, we set $M = 110$ in our simulations.

All experiments performed in this section are based on a number of one-thousand Monte-Carlo runs and the standard deviation α of the measurement noise is set to 1. The objective of the conducted experiments is to test the estimation power of our formula (8) against different choices of the diffusion coefficient σ and of the mean reversion speed κ. Clearly, we expect better results when σ is small since the formula has been derived under the approximation that σ is zero. Moreover, the introduced approximation makes the expected energy lifetime independent of κ, but in general one expects the mean reversion speed κ to affect the expected sensor lifetime, especially for large values of the diffusion coefficient σ. As κ gets smaller, the effect of the introduced approximation gets reduced. In the limiting case when $\kappa = 0$, $X_t^{(2)}$ degenerates to a Wiener process, and our formula becomes exact. This can be confirmed from Figures 3 and 4, where an extremely good match is observed for the all range of σ and γ values. In case when $\kappa \neq 0$, then we expect our formula to be a good predictor of interarrival times for small σ. We ran two experiments, only differing in the choice of the mean reversion speed parameter κ, which is set to 1 for the first and to 3 for the second experiment.

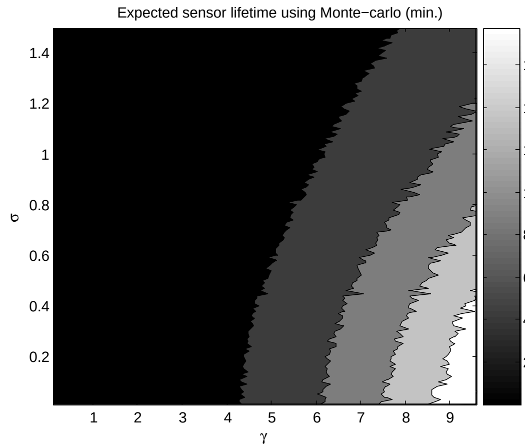

Figure 3: Expected sensor lifetime estimated using one-thousand Monte-carlo runs. The mean reversion speed is set to $\kappa = 0$. The time is measured in minutes

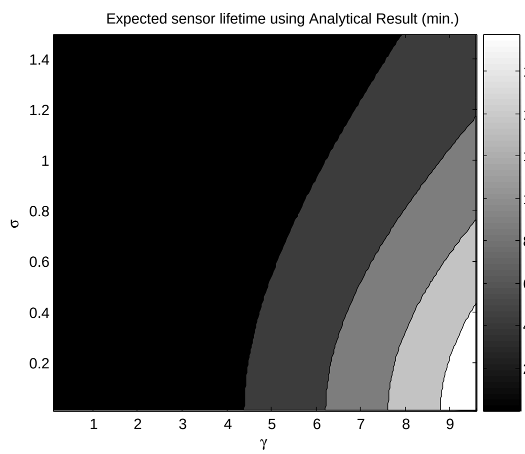

Figure 4: Expected sensor lifetime predicted using formula (8). The mean reversion speed is set to $\kappa = 0$. The time is measured in minutes.

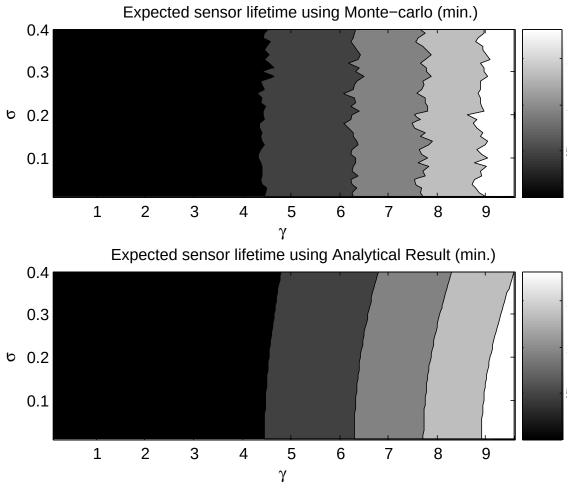

Figure 5: Expected sensor lifetime. The mean reversion speed is set to $\kappa = 1$, $\sigma \in [0.01, 0.4]$. The time is measured in minutes.

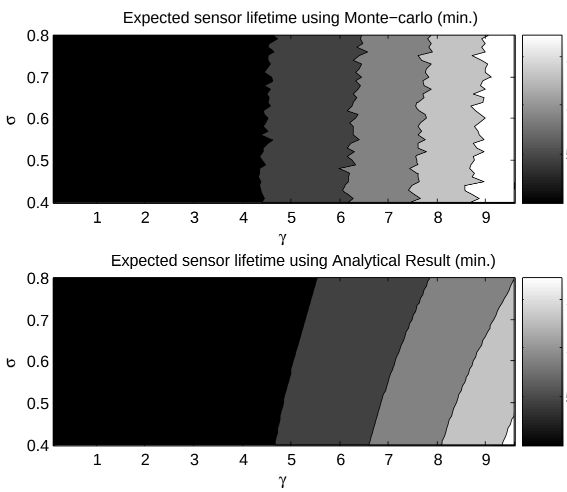

Figure 6: Expected sensor lifetime. The mean reversion speed is set to $\kappa = 1$, $\sigma \in [0.4, 0.8]$. The time is measured in minutes.

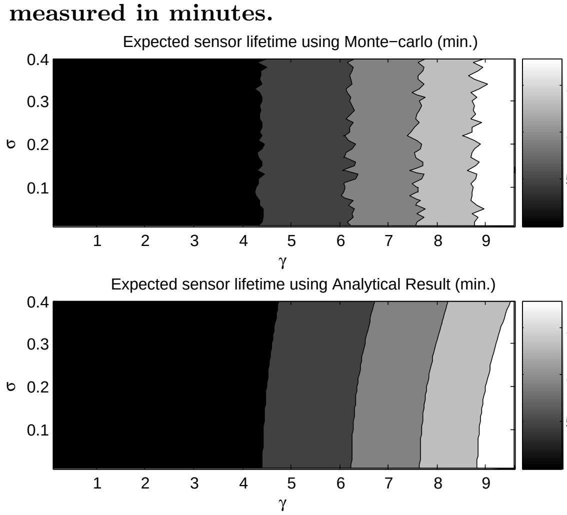

Figure 7: Expected sensor lifetime. The mean reversion speed is set to $\kappa = 3$, $\sigma \in [0.01, 0.4]$. The time is measured in minutes.

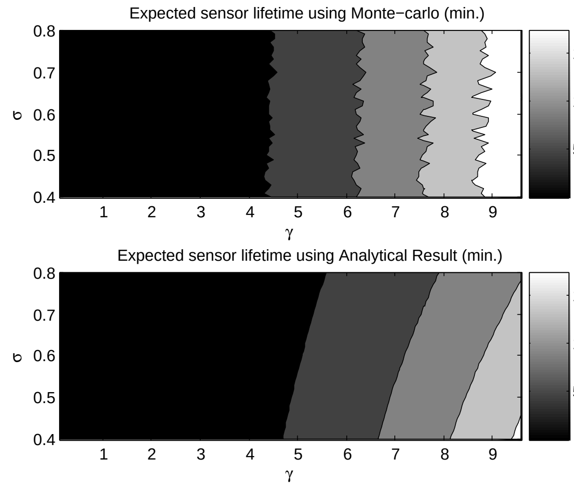

Figure 8: Expected sensor lifetime. The mean reversion speed is set to $\kappa = 3$, $\sigma \in [0.4, 0.8]$. The time is measured in minutes.

We consider values of σ ranging in $[0.01, 0.8]$ with a step size of 0.01, and values of γ ranging in $[0.1, 10]$ with step size 0.5.

The results are reported in Figures 5-8. We can see from the graphs that the Monte-carlo estimates agree very nicely with the values predicted by the theoretical formula when $\sigma < 0.4$. In case when σ gets larger, then the theoretical formula tends to overestimate the expected sensor lifetime, and the overestimation gets larger as the size of the region γ increases. This comes as no surprise since the validity of the formula is based on small values of the diffusion coefficient σ. It may be argued that increasing the mean reversion speed κ makes the process revert faster towards zero, thus increasing the time needed to touch the boundary of the region, and our formula does not account for that. This adds evidence to the fact that our formula tends to overestimate the expected sensor lifetime. However, a closer look at the Monte-carlo estimates reported in the figures show that the expected sensor lifetime is not much sensitive to κ.

5. CONCLUSIONS

In this paper we have studied the problem of estimating the expected sensor lifetime in event based control systems. We have presented a study case consisting of a monitoring sensor which sends signals only when significant deviations from expected behavior occur. We have derived an analytical formula to predict approximately the average sensor lifetime under the assumption that the model deviation and the noise of the measuring device are respectively an Ornstein-Uhlenbeck and Wiener process. We have verified by means of Monte-carlo runs that the introduced approximation does not impact the estimation accuracy of the expected sensor lifetime when the diffusion coefficient of the process representing the model deviation is not too large. Furthermore, our formula becomes exact as the mean reversion speed goes to zero. In a future continuation of the work, we would like to study the relation between the size of the monitoring region and the state estimation error introduced when using signals received at those random times to perform the estimation. We are also interested in extending the current work to a wireless sensor network consisting of a larger number of sensor nodes. In addition to some appropriate sensor scheduling schemes, we would like to show how such event based scheme can help increase the overall network lifetime.

Acknowledgments

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APPENDIX

A. THE GENERATOR

An Ito diffusion process has the general form

$$d\boldsymbol{X}_t = \boldsymbol{b}(\boldsymbol{X}_t)dt + \hat{\sigma}(\boldsymbol{X}_t)d\boldsymbol{W}_t \tag{19}$$

where $\boldsymbol{X} \in \mathbb{R}^d$, $\boldsymbol{b} : \mathbb{R}^d \to \mathbb{R}^d$, $\hat{\sigma} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$, and $\boldsymbol{W}_t$ is an $\mathbb{R}^d$-valued Wiener process. The generator of the process [4] is a linear operator of the form

$$\widehat{\mathcal{L}} = \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{d} \hat{\sigma}_{ik}\hat{\sigma}_{jk}\frac{\partial^2}{\partial x_i \partial x_j} \tag{20}$$

For our Ito diffusion process given by eq.(2) and (3) we have:

$$d = 2 \quad \boldsymbol{b}(\boldsymbol{X}_t) = \begin{pmatrix} 0 \\ -\kappa X_t^{(2)} \end{pmatrix} \quad \hat{\sigma}(\boldsymbol{X}_t) = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma \end{pmatrix} \tag{21}$$

Therefore, applying the formula given by eq.(11) we obtain that the generator on $\mathbb{R}^2$ is

$$\widehat{\mathcal{L}} := -\kappa x_2 \partial_{x_2} + \frac{1}{2}\left(\alpha^2 \partial^2_{x_1 x_1} + \sigma^2 \partial^2_{x_2 x_2}\right) \tag{22}$$

B. THE DYNKIN'S LEMMA

We use a result from stochastic analysis known as Dynkin's lemma, and stating:

LEMMA 2 ([8]). *Let $\mathbf{X}_t$ be a diffusion process and f be a smooth function. Suppose τ is a stopping time such that $E[\tau] < \infty$. Then*

$$E[f(\mathbf{X}^{\mathbf{x}}_\tau)] = f(\mathbf{x}) + E\left[\int_0^\tau \mathcal{L}f(X^{\mathbf{x}}_s)ds\right] \tag{23}$$

where $\mathbf{X}^{\mathbf{x}}_t$ is a process starting at $\mathbf{x}$ at time 0 and $\mathcal{L}$ is the generator of the diffusion process $\mathbf{X}$.

C. THE EXPECTED INTERARRIVAL TIME

The generator the diffusion process in eq.(11) is given by

$$\hat{\mathcal{L}} = \frac{1}{2}[\alpha^2 \partial^2_{x_1 x_1} + \sigma^2 \partial^2_{x_2 x_2}] - \kappa x_2 \partial_{x_2} \tag{24}$$

We need to solve the boundary value problem $\hat{\mathcal{L}}f = 1$ in the strip $\Omega := \{(x_1, x_2) : |x_1 + x_2| < \gamma\}$ (zero boundary conditions). To make the problem tractable, we introduce the following change of coordinate:

$$y_1 = \frac{x_1 + x_2}{\sqrt{1+\sigma^2}} \quad y_2 = \frac{\sigma x_1 - \frac{x_2}{\sigma}}{\sqrt{1+\sigma^2}} \tag{25}$$

We then have that

$$\partial_{x_1 x_1} = \frac{\alpha^2}{1+\sigma^2}(\partial_{y_1 y_1} + 2\sigma\partial_{y_1 y_2} + \sigma^2 \partial_{y_2 y_2}) \tag{26}$$

$$\partial_{x_2 x_2} = \frac{1}{1+\sigma^2}(\sigma^2 \partial_{y_1 y_1} - 2\sigma\partial_{y_1 y_2} + \partial_{y_2 y_2}) \tag{27}$$

$$\partial_{x_2} = \frac{1}{\sqrt{1+\sigma^2}}\partial_{y_1} - \frac{1}{\sigma\sqrt{1+\sigma^2}}\partial_{y_2} \tag{28}$$

and from eq.(25) we have that

$$x_2 = \frac{\sigma}{\sqrt{1+\sigma^2}}(\sigma y_1 - y_2) \tag{29}$$

Therefore, the generator 24 can be rewritten in the (x, y) system of coordinates as

$$\begin{aligned}\hat{\mathcal{L}} = {} & \frac{(\alpha^2+\sigma^2)\partial^2_{y_1 y_1} + 2\sigma(\alpha - 1)\partial^2_{y_1 y_2} + (\alpha\sigma^2 + 1)\partial^2_{y_2 y_2}}{2(1+\sigma^2)} \\ & -\kappa\frac{\sigma^2 y_1 \partial_{y_1} - \sigma(y_1\partial_{y_2} - y_2\partial_{y_1}) + y_2\partial_{y_2}}{1+\sigma^2}\end{aligned} \tag{30}$$

and the region becomes

$$\Omega = \{(y_1, y_2) : |y_1| < \frac{\gamma}{\sqrt{1+\sigma^2}}\} \tag{31}$$

Regular perturbation theory may now be applied to this problem. For the lowest order approximation, obtained setting σ to zero, we get that

$$f(y_1, y_2) = \frac{1}{\alpha^2}y_1^2 - \frac{\gamma^2}{\alpha^2(1+\sigma^2)} \tag{32}$$

is a solution of the boundary value problem.